arXiv:0910.0428v4 [math.NT] 20 Oct 2009

VERY SMALL INTERVALS CONTAINING AT LEAST THREE PRIMES

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ABSTRACT. Let p_n is the n-th prime. With help of the Cramér-like model, we prove that the set of intervals of the form $(2p_n,\ 2p_{n+1})$ containing at list 3 primes has a positive density with respect to the set of all intervals of such form.

1. Introduction

Everywhere below we understand that p_n is the n-th prime and $\mathbb{P}$ is the class of all increasing infinite sequences of primes. If $A \in \mathbb{P}$ then we denote $\mathcal{A}$ the event that prime p is in A. In particular, an important role in our constructions play the following sequences from $\mathbb{P}:\ \ A_i$ is the sequence of those primes p_k, for which the interval $(2p_k, 2p_{k+1})$ contains at least i primes, $i = 1, 2, ...$ By $\mathcal{A}_i(n)$, we denote the event that p_n is in A_i, $i = 1, 2, ...$

In [1] we considered the following problem. Let p be an odd prime. Let, furthermore, $p_n < p/2 < p_{n+1}$. According to the Bertrand's postulate, between $p/2$ and p there exists a prime. Therefore, $p_{n+1} \le p$. Again, by the Bertrand's postulate, between p and $2p$ there exists a prime. More subtle question is the following.

Problem 1. *Consider the sequence S of primes p possessing the property: if $p/2$ lies in the interval $(p_n,\ p_{n+1})$ then there exists a prime in the interval $(p, 2p_{n+1})$. With what probability a random prime q belongs to S (or the event $\mathcal{S}$ does occur)?.*

In this paper we prove the following theorem.

Theorem 1. *The set of intervals of the form $(2p_n,\ 2p_{n+1})$ containing at list 3 primes has a positive density with respect to the set of all intervals of such form.*

2. Criterions for **R**-primes, **L**-primes and **RL**-primes

In [1] we found a sieve for the separating **R**-primes from all primes and shown how to receive the corresponding sieve for **L**-primes. Now we give simple criterions for them.

1991 *Mathematics Subject Classification.* Primary 11A41, secondary 11B05.

Theorem 2. *1) p_n is* **R***-prime if and only if* $\pi(\frac{p_n}{2}) = \pi(\frac{p_{n+1}}{2})$;
2) p_n is **L***-prime if and only if* $\pi(\frac{p_n}{2}) = \pi(\frac{p_{n-1}}{2})$;
3) p_n is **RL***-prime if and only if* $\pi(\frac{p_{n-1}}{2}) = \pi(\frac{p_{n+1}}{2})$.

Proof. 1) Let $\pi(\frac{p_n}{2}) = \pi(\frac{p_{n+1}}{2})$ is valid. Now if $p_k < p_n/2 < p_{k+1}$, and between $p_n/2$ and $p_{n+1}/2$ do not exist primes. Thus $p_{n+1}/2 < p_{k+1}$ as well. Therefore, we have $2p_k < p_n < p_{n+1} < 2p_{k+1}$, i.e. p_n is **R**-prime. Conversely, if p_n is **R**-prime, then $2p_k < p_n < p_{n+1} < 2p_{k+1}$, and $\pi(\frac{p_n}{2}) = \pi(\frac{p_{n+1}}{2})$ is valid. 2) is proved quite analogously and 3) follows from 1) and 2). ■

3. Proof of a "precise symmetry" conjecture

We start with a proof of the following conjecture [1].

Conjecture 1. *Let* $\mathbf{R}_n$ $(\mathbf{L}_n)$ *denote the n-th term of the sequence* **R** (**L**). *Then we have*

$$\mathbf{R}_1 \le \mathbf{L}_1 \le \mathbf{R}_2 \le \mathbf{L}_2 \le ... \le \mathbf{R}_n \le \mathbf{L}_n \le ... \tag{3.1}$$

Proof of Conjecture 1. It is clear that the intervals of considered form, containing not more than one prime, contain neither **R**-primes nor **L**-primes. Moving such intervals, consider the first from the remaining ones. The first its prime is an **R**-prime ($\mathbf{R}_1$). If it has only two primes, then the second prime is an **L**-prime ($\mathbf{L}_1$), and we see that $(\mathbf{R}_1) < (\mathbf{L}_1)$; on the other hand if it has k primes, then beginning with the second one and up to the $(k-1)$-th we have **RL**-primes, i.e. primes which are simultaneously **R**-primes and **L**-primes. Thus, taking into account that the last prime is only **L**-prime , we have

$$\mathbf{R}_1 < \mathbf{L}_1 = \mathbf{R}_2 = \mathbf{L}_2 = \mathbf{R}_3 = ... = \mathbf{L}_{k-1} = \mathbf{R}_{k-1} < \mathbf{L}_k.$$

The second remaining interval begins with an **R**-prime and the process repeats. ■

Remark 1. *Note that a corollary that "the number of* **RL***-primes not exceeding x is not less than the number of A_3-primes not exceeding x" is absolutely erroneously. Indeed, we should take into account that every interval of the form $(2p_n,\ 2p_{n+1})$ containing* **RL***-prime contains* at least *3 primes not exceeding x. A right corollary is the following. Since, by the condition of Problem 1, a prime p already lies in a interval $(2p_n,\ 2p_{n+1})$, then we should consider* only *intervals containing at least prime. Denote $\mathcal{A}_k,\ k = 1, ...,$ the event that a random interval $(2p_n,\ 2p_{n+1})$ contains at least $k,\ 1, 2, ...$* primes. If $P(\mathcal{A}_1) = q$, then we have

$$P(\mathcal{A}_k) = q^k, \quad k = 1, 2, ... \tag{3.2}$$

Let, furthermore, $\mathcal{A}^{(k)}, \quad k = 1, ...,$ the event that a random interval $(2p_n, \quad 2p_{n+1})$ contains *exact* $k, \quad 1, 2, ...$ primes. Then, by (3.2),

$$P(\mathcal{A}^{(k)} = P(\mathcal{A}_k) - P(\mathcal{A}_{k+1}) = (q-1)q^k, \quad k = 1, 2, ...$$

and we have

$$P(\mathbf{RL}) = (1-q)\sum_{k \ge 3} \frac{k-2}{k} q^{k-1} = 2 - q + 2\frac{1-q}{q}\ln(1-q). \tag{3.3}$$

4. Proof of Theorem 1

The theorem immediately follows from the positivity of probability $P(\mathbf{RL})$. In fact, in [1] we proved that $q \approx 0.8010$ and $P(\mathbf{RL}) \approx 0.3980$. ■
Note that by the Cramér's 1937 conjecture $2p_{n+1} - 2p_n < (2+\varepsilon)\ln^2 n$. Thus, there exists an infinite sequence of the intervals of such small length, but having at least three primes, and, moreover, this sequence has a positive density with respect to the sequence of all intervals of the form $(2p_n, \quad 2p_{n+1})$. By this way, in view of (3.2), it could be proved a more general result.

Theorem 3. *Let h be arbitrary large but a fixed positive integer. Then the set of intervals of the form $(2p_n, \quad 2p_{n+1})$ containing at list h primes has a positive density with respect to the set of all intervals of such form.*

Quite analogously one can consider an m-generalization of Theorem 1 for every $1 < m < 2$. Here the case of especial interest is the case of the values of m close to 1.

References

[1] . V. Shevelev *Three probabilities concerning prime gaps* http://arxiv.org/abs/0909.0715
[2] . V. Shevelev *Critical small intervals containing primes*http://arxiv.org/abs/0908.2319

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